Geometric properties of logarithmic limit sets over the reals

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Logarithmic limit sets of complex algebraic sets have been first studied in [4], and they were further studied by many people, for example in [7]. The logarithmic limit set of a complex algebraic set is a polyhedral complex of the same dimension as the algebraic set, it is described by tropical equations and it is the image, under the component-wise valuation map, of an algebraic set over an algebraically closed non-archimedean field. We can extend these properties to the real case, see [1] for details.

Let $V \subset (\mathbb{R}_{>0})^n$ be a real semi-algebraic set. We apply the **Maslov dequantization** to V: for $t \in (0,1)$ the **amoeba** of V is

$$\mathcal{A}_t(V) = \{(\log_{\left(\frac{1}{t}\right)}(x_1), \ldots, \log_{\left(\frac{1}{t}\right)}(x_n)) \mid (x_1, \ldots, x_n) \in V\}$$

the **logarithmic limit set** is the limit of the amoebas

$$\mathcal{A}_0(V) = \lim_{t \to 0} \mathcal{A}_t(V)$$

To study the set $\mathcal{A}_0(V)$ we use the following property: the point $(0, \ldots, 0, -1)$ is in $\mathcal{A}_0(V)$ if and only if there exists a sequence $(x_k) \subset V$ and $a_1, \ldots, a_{n-1} > 0$ such that $(x_k) \to (a_1, \ldots, a_{n-1}, 0)$. This proposition shows that the special point $(0, \ldots, 0, -1)$ is particularly easy to control. If we want to study another point $x \in \mathbb{R}^n$, we can act on $\mathbb{R}^n$ by linear maps, moving x to the special point.

Let $B = (b_{ij}) \in GL_n(\mathbb{R})$, then B acts linearly on $\mathbb{R}^n$. By conjugation with the componentwise logarithm map, B acts on $(\mathbb{R}_{>0})^n$:

$$\overline{B}(x) = (x_1^{b_{11}} x_2^{b_{12}} \cdots x_n^{b_{1n}}\ ,\ \ldots\ ,\ x_1^{b_{n1}} x_2^{b_{n2}} \cdots x_n^{b_{nn}})$$

If $V \subset (\mathbb{R}_{>0})^n$, then $B(\mathcal{A}_0(V))$ is the logarithmic limit set of $\overline{B}(V)$. Anyway, if the entries of B are not rational, $\overline{B}(V)$ is not semi-algebraic. The category of semi-algebraic sets is too small for our methods.

We need to work in a more general setting: sets definable in an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. For example the structure $\mathcal{OS}^{\mathbb{R}}$ of real closed field expanded with all the power functions is o-minimal, polynomially bounded, with field of exponents $\mathbb{R}$ (see [6]).

Theorem 1. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. Then the logarithmic limit set $\mathcal{A}_0(V) \subset \mathbb{R}^n$ is a polyhedral cone, and $\dim \mathcal{A}_0(V) \leq \dim V$.*

In the real case, the behavior of logarithmic limit sets is less regular than the behavior they have in the complex case. It is easy to show examples where $\dim \mathcal{A}_0(V) < \dim V$, and where $\mathcal{A}_0(V)$ is not equidimensional. Also the combinatorics is not well understood.

Let $V \subset (\mathbb{C}^*)^n$ be an algebraic hypersurface with real equation f, and let $V_{>0} = V \cap (\mathbb{R}_{>0})^n$, its positive part. Then $\mathcal{A}_0(V)$ is a polyhedral fan, dual to the Newton polytope of f. The set $\mathcal{A}_0(V_{>0})$ is a subset of $\mathcal{A}_0(V)$, a polyhedral complex, but it is not always a subcomplex. For example, consider the "Cartan umbrella" $V = \{(x,y,z) \in (\mathbb{C}^*)^3 \mid x^2(1-(z-2)^2) = x^4+(y-1)^2\}$. Then $\mathcal{A}_0(V_{>0})$ is only the ray in the direction $(-1,0,0)$, but this set is in the interior of a face of the dual fan of $f = x^4 + x^2(z^2 - 4z + 3) + y^2 - 2y + 1$.

Let S be an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. The Hardy field can be defined as the set of germs of definable functions of one variable:

$$H(S) = \{(f,\varepsilon) \mid f : (0,\varepsilon) \longrightarrow \mathbb{R} \text{ definable } \}/\sim$$

$$(f,\varepsilon) \sim (g,\varepsilon') \Leftrightarrow \exists\delta > 0 : f_{|(0,\delta)} = g_{|(0,\delta)}$$

The set $H(S)$ inherit an S-structure from $\mathbb{R}$, that is an elementary extension. In particular $H(S)$ is a non-archimedean real closed field, with a surjective real valuation.

Let $W \subset (H(S)_{>0})^n$ be a definable set. We define the Log map as:

$$\text{Log} : (H(S)_{>0})^n \ni (x_1,\ldots,x_n) \longrightarrow (-v(x_1),\ldots,-v(x_n))$$

Theorem 2. *The set* $\text{Log}(W) \subset \mathbb{R}^n$ *is a polyhedral complex, and* $\dim(\text{Log}(W)) \leq \dim(W)$.

If W is a semi-linear set (a polyhedron) these objects were studied in [5]. These objects are very similar to the Positive Tropical Varieties (see [8]) but there are examples where they differs.

Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set in S, and let $\overline{V} \subset (H(S)_{>0})^n$ be its extension to the Hardy field. We have

Theorem 3.

$$\lim_{t\to 0} \mathcal{A}_t(V) = \mathcal{A}_0(V) = \text{Log}(\overline{V})$$

In the general case, we can construct a family of definable sets $V_t \subset (\mathbb{R}_{>0})^n$ such that

$$\lim_{t\to 0} \mathcal{A}_t(V_t) = \text{Log}(W)$$

this construction is a generalization of the patchworking families, see [9]. If W is defined by a first order formula ϕ:

$$W = \{(x_1,\ldots,x_n) \mid \phi(x_1,\ldots,x_n,f_1,\ldots,f_m)\}$$

where $f_1,\ldots,f_m \in H(S)$ are germs of definable functions, then

$$V_t = \{(x_1,\ldots,x_n) \in (\mathbb{R}_{>0})^n \mid \phi(x_1,\ldots,x_n,f_1(t),\ldots,f_m(t))\}$$

Finally, we can describe logarithmic limit sets with tropical equations. A positive formula in the symbols $\mathcal{OS}^{\mathbb{R}}$ is a first order formula containing only the connectives $\vee$ and $\wedge$ and the quantifiers $\forall$, $\exists$. (No $\neg, \Rightarrow, \Leftrightarrow$). Every subset of $(\mathbb{R}_{>0})^n$ that is defined by a quantifier-free positive formula is closed. Every closed semi-algebraic set is defined by a positive quantifier-free formula.

Theorem 4. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable by a positive formula in the symbols $\mathcal{OS}^{\mathbb{R}}$ with parameters in $\mathbb{R}_{>0}$. Note that every closed semi-algebraic set satisfies the hypothesis. Then there exists a positive formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and parameter $a_1, \ldots, a_m \in \mathbb{R}_{>0}$ such that*

$$V = \{x \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

$$\mathcal{A}_0(V) = \{x \mid \phi_{\mathbb{T}}(x_1, \ldots, x_n, 0, \ldots, 0)\}$$

Where $\phi_{\mathbb{T}}$ is the formula ϕ where the operations are interpreted tropically, i.e. "+" becomes "max" and "·" becomes "+".

Our motivations for this work come from low-dimensional topology, see [2] and [3]. Let $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ denote parameter space of convex real projective structures on a closed orientable n-manifold M such that $\pi_1(M)$ is torsion free, virtually centerless and Gromov hyperbolic (for example M can be every hyperbolic manifold whose fundamental group is torsion-free). The space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ can be identified with a closed semi-algebraic subset of the character variety $\mathrm{Char}(\pi_1(M), SL_{n+1}(\mathbb{R}))$. We can construct compactifications of semi-algebraic sets using inverse systems of logarithmic limit sets. The boundary points are tropical images of the extension of the semi-algebraic set to a real closed non-archimedean field $\mathbb{F}$ with a surjective real valuation. In particular the points of $\partial\mathcal{T}^c_{\mathbb{RP}^n}(M)$ are the tropical images of elements of $\mathrm{Char}(\pi_1(M), SL_{n+1}(\mathbb{F}))$, where $\mathbb{F}$ is as above. Using this fact we can give a geometric interpretation of the boundary points of $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ as actions on "tropical projective spaces", constructed using a generalization of the Bruhat-Tits buildings for $SL_{n+1}(\mathbb{F})$.

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